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Class 12 - Mathematics

Chapter 1: RELATIONS AND FUNCTIONS

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📖 Chapter Overview

Chapter: Relations and Functions

Class: XII - CBSE Board

Academic Year: 2025-26

Weightage: 10 Marks (Approx.)

Difficulty Level: Moderate to High

Important Topics: Equivalence Relations, One-One & Onto Functions, Composition, Invertible Functions

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1. INTRODUCTION

The concept of **relations** and **functions** is fundamental in mathematics and has applications across various fields including physics, engineering, computer science, and economics.

Relation

Let A and B be two non-empty sets. A **relation R** from set A to set B is a subset of the Cartesian product A × B.

$$R \subseteq A \times B$$

If (a, b) ∈ R, we say that **a is related to b** under the relation R, written as **a R b**.

Relation on a Set

A relation in a set A is a subset of A × A. This is called a **relation on set A**.

⚠️ **IMPORTANT:** Functions are special types of relations where each element of the domain is related to exactly one element of the codomain.

2. TYPES OF RELATIONS

2.1 Empty Relation

Empty Relation

A relation R in a set A is called an **empty relation** if no element of A is related to any element of A.

$$R = \varphi \subset A \times A$$

Example:

In the set A = {1, 2, 3, 4}, the relation R = {(a, b) : a - b = 10} is an empty relation because no pair (a, b) from A × A satisfies the condition a - b = 10.

2.2 Universal Relation

Universal Relation

A relation R in a set A is called a **universal relation** if each element of A is related to every element of A.

$$R = A \times A$$

Example:

In the set A = {1, 2, 3}, the relation R = {(a, b) : |a - b| ≥ 0} is universal because all pairs (a, b) from A × A satisfy this condition.

⚠️ **IMPORTANT:** Both empty and universal relations are called **trivial relations**.

2.3 Reflexive Relation

Reflexive Relation

A relation R in a set A is called **reflexive** if (a, a) ∈ R for every a ∈ A.

$$\forall\, a \in A,\ (a, a) \in R$$

In simple words: Every element is related to itself.

Example:

The relation R = {(a, b) : a ≤ b} in the set of real numbers is reflexive because a ≤ a for all real numbers a.

2.4 Symmetric Relation

Symmetric Relation

A relation R in a set A is called **symmetric** if (a, b) ∈ R implies (b, a) ∈ R for all a, b ∈ A.

$$(a, b) \in R \Rightarrow (b, a) \in R$$

In simple words: If a is related to b, then b is related to a.

Example:

The relation R = {(a, b) : a - b is even} in the set of integers is symmetric because if a - b is even, then b - a = -(a - b) is also even.

2.5 Transitive Relation

Transitive Relation

A relation R in a set A is called **transitive** if (a, b) ∈ R and (b, c) ∈ R implies (a, c) ∈ R for all a, b, c ∈ A.

$$(a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R$$

In simple words: If a is related to b and b is related to c, then a is related to c.

Example:

The relation R = {(a, b) : a ≤ b} in the set of real numbers is transitive because if a ≤ b and b ≤ c, then a ≤ c.

3. EQUIVALENCE RELATIONS

Equivalence Relation

A relation R in a set A is called an **equivalence relation** if R is:

1. **Reflexive:** (a, a) ∈ R for all a ∈ A
2. **Symmetric:** (a, b) ∈ R ⇒ (b, a) ∈ R
3. **Transitive:** (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R

R is Equivalence ⟺ R is Reflexive AND Symmetric AND Transitive

✅ **How to Prove Equivalence Relation (Board Exam Pattern):**

- **Step 1:** Prove Reflexive - Show (a, a) ∈ R for all a ∈ A
- **Step 2:** Prove Symmetric - Assume (a, b) ∈ R and prove (b, a) ∈ R
- **Step 3:** Prove Transitive - Assume (a, b) ∈ R and (b, c) ∈ R, then prove (a, c) ∈ R
- **Step 4:** Conclude that R is an equivalence relation

Example 1: Standard Equivalence Relation

Question: Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides a - b} is an equivalence relation.

Solution:

Step 1: Checking Reflexive

For any a ∈ Z, we have a - a = 0.

Since 2 divides 0, we have (a, a) ∈ R for all a ∈ Z.

∴ **R is reflexive.**

Step 2: Checking Symmetric

Let $(a, b) \in R$

$\Rightarrow$ 2 divides $(a - b)$

$\Rightarrow$ 2 divides $-(a - b) = (b - a)$

$\Rightarrow (b, a) \in R$

∴ **R is symmetric.**

Step 3: Checking Transitive

Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ 2 divides $(a - b)$ and 2 divides $(b - c)$

$\Rightarrow$ 2 divides $(a - b) + (b - c) = (a - c)$

$\Rightarrow (a, c) \in R$

∴ **R is transitive.**

> **Conclusion:** Since R is reflexive, symmetric, and transitive, R is an equivalence relation. ✓

3.1 Equivalence Classes

Equivalence Class

Let R be an equivalence relation in set A. For any element $a \in A$, the **equivalence class** **[a]** is defined as:

$$[a] = \{x \in A : (x, a) \in R\} = \{x \in A : x R a\}$$

The equivalence class [a] is the set of all elements that are related to a.

Properties of Equivalence Classes:

- All elements in an equivalence class are related to each other
- No element of one equivalence class is related to any element of another equivalence class
- The union of all equivalence classes equals the original set A
- Any two equivalence classes are either equal or disjoint
- Equivalence classes partition the set A into disjoint subsets

Example 2: Equivalence Relation with Classes

Question: Let R be the relation in set A = {1, 2, 3, 4, 5, 6, 7} given by R = {(a, b) : both a and b are either odd or even}. Show that R is an equivalence relation. Further, show that all elements of {1, 3, 5, 7} are related to each other and all elements of {2, 4, 6} are related to each other.

Solution:

Step 1: Checking Reflexive

For any a ∈ A, both a and a have the same parity (both odd or both even).

Therefore, (a, a) ∈ R for all a ∈ A.

∴ **R is reflexive.**

Step 2: Checking Symmetric

Let (a, b) ∈ R

⇒ Both a and b are either odd or even

⇒ Both b and a are either odd or even

⇒ (b, a) ∈ R

∴ **R is symmetric.**

Step 3: Checking Transitive

Let (a, b) ∈ R and (b, c) ∈ R

⇒ a, b have the same parity and b, c have the same parity

⇒ a, b, c all have the same parity

⇒ (a, c) ∈ R

∴ **R is transitive.**

> **Conclusion:** R is an equivalence relation. ✓

Equivalence Classes:

• **[1] = [3] = [5] = [7] = {1, 3, 5, 7}** (all odd numbers)

• **[2] = [4] = [6] = {2, 4, 6}** (all even numbers)

• All elements of {1, 3, 5, 7} are related to each other (all odd)

• All elements of {2, 4, 6} are related to each other (all even)

• No element from {1, 3, 5, 7} is related to any element from {2, 4, 6}

4. TYPES OF FUNCTIONS

4.1 One-One Function (Injective)

One-One Function (Injective)

A function $f : X \rightarrow Y$ is called **one-one** (or **injective**) if different elements of X have different images in Y.

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

for all $x_1, x_2 \in X$

In other words: No two different elements of X have the same image in Y.

✅ **How to Prove One-One:**

- Assume $f(x_1) = f(x_2)$
- Algebraically simplify to show $x_1 = x_2$
- Conclude that f is one-one

⚠️ **IMPORTANT:** A function is **NOT one-one (many-one)** if there exist $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

4.2 Onto Function (Surjective)

Onto Function (Surjective)

A function $f : X \rightarrow Y$ is called **onto** (or **surjective**) if every element of Y is the image of some element of X.

For every y ∈ Y, there exists x ∈ X such that f(x) = y

In other words: Range of f = Co-domain of f = Y

✅ **How to Prove Onto:**

- Take an arbitrary element y in the co-domain Y
- Solve f(x) = y for x
- Show that the value of x obtained belongs to the domain X
- Conclude that f is onto

⚠️ **IMPORTANT:** A function is **NOT onto** if there exists at least one element y ∈ Y which is not the image of any element in X.

4.3 Bijective Function

Bijective Function

A function f : X → Y is called **bijective** if it is both one-one and onto.

f is Bijective ⟺ f is One-One AND Onto

Summary Table - Types of Functions:

Function Type	Property	Mathematical Condition

One-One	Different inputs → Different outputs	$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
Onto	Every element in codomain has pre-image	Range = Co-domain
Bijective	Both One-One and Onto	Perfect one-to-one correspondence

Example 3: Proving One-One and Onto

Question: Show that $f : R \to R$ given by $f(x) = 2x$ is one-one and onto.

Solution:

Checking One-One:

Let $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

∴ f is one-one. ✓

Checking Onto:

Let $y \in R$ (co-domain). We need to find $x \in R$ such that $f(x) = y$.

Let $f(x) = y$

$\Rightarrow 2x = y$

$\Rightarrow x = y/2$

Since $y \in R$, we have $x = y/2 \in R$ (domain).

For this x, $f(x) = 2(y/2) = y$.

∴ f is onto. ✓

> **Conclusion:** f is both one-one and onto, hence bijective. ✓

Example 4: One-One but Not Onto

Question: Show that $f : N \to N$ given by $f(x) = 2x$ is one-one but not onto.

Solution:

Checking One-One:

Let $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

∴ f is one-one. ✓

Checking Onto:

Consider $1 \in N$ (co-domain).

For $f(x) = 1$, we need $2x = 1$

$\Rightarrow x = 1/2$

But $1/2 \notin N$ (domain).

Therefore, 1 has no pre-image in N.

∴ f is NOT onto. ✗

Conclusion: f is one-one but not onto.

5. COMPOSITION OF FUNCTIONS

Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two functions. The **composition of f and g**, denoted by **gof** (read as "g composition f" or "g circle f"), is defined as:

$$gof : A \to C$$

$$(gof)(x) = g(f(x)) \text{ for all } x \in A$$

⚠️ **IMPORTANT:** For gof to be defined, the range of f must be a subset of the domain of g. The composition gof means: first apply f, then apply g (read from right to left).

📌 **Properties of Composition:**

- **Not Commutative:** In general, $gof \neq fog$
- **Associative:** $(hog)of = ho(gof)$ when defined
- If f and g are one-one, then gof is one-one
- If f and g are onto, then gof is onto
- If f and g are bijective, then gof is bijective

Example 5: Composition of Functions

Question: Find gof and fog if $f : R \to R$ and $g : R \to R$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution:

Finding gof:

$(gof)(x) = g(f(x))$

$= g(\cos x)$

$= 3(\cos x)^2$

$= \mathbf{3\cos^2 x}$

Finding fog:

$(fog)(x) = f(g(x))$

$= f(3x^2)$

$= \mathbf{\cos(3x^2)}$

Checking if gof = fog:

At $x = 0$:

- $(gof)(0) = 3\cos^2(0) = 3(1)^2 = 3$
- $(fog)(0) = \cos(3\cdot0^2) = \cos(0) = 1$

Since $3 \neq 1$, we have **gof ≠ fog**. ✓

6. INVERTIBLE FUNCTIONS

Invertible Function

A function f : X → Y is called **invertible** if there exists a function g : Y → X such that:

$$gof = I_X \text{ and } fog = I_Y$$

where I_X and I_Y are identity functions on X and Y respectively.

The function g is called the **inverse of f** and is denoted by $\mathbf{f^{-1}}$.

⭐ FUNDAMENTAL THEOREM

A function f : X → Y is invertible if and only if f is bijective (one-one and onto).

✅ How to Find Inverse Function:

- **Step 1:** Verify that f is bijective (one-one and onto)
- **Step 2:** Let y = f(x)
- **Step 3:** Solve for x in terms of y
- **Step 4:** Replace y with x to get $f^{-1}(x)$
- **Step 5:** Verify: $fof^{-1} = I$ and $f^{-1}of = I$

⚠️ **IMPORTANT:** If f is invertible, then f^{-1} is unique. Also, $(f^{-1})^{-1} = f$.

Example 6: Finding Inverse Function

Question: Let f : R → R be defined as f(x) = 2x. Show that f is invertible and find f^{-1}.

Solution:

Step 1: Check if f is bijective

We have already proved in Example 3 that f(x) = 2x is both one-one and onto.

Therefore, f is bijective, hence invertible. ✓

Step 2: Find f^{-1}

Let y = f(x)

⇒ y = 2x

⇒ x = y/2

Therefore, $\mathbf{f^{-1}(y) = y/2}$

or $\mathbf{f^{-1}(x) = x/2}$

Step 3: Verification

Checking fof^{-1} = I:

$(fof^{-1})(x) = f(f^{-1}(x)) = f(x/2) = 2(x/2) = x$ ✓

Checking $f^{-1}of$ = I:

$(f^{-1}of)(x) = f^{-1}(f(x)) = f^{-1}(2x) = (2x)/2 = x$ ✓

Answer: $f^{-1}(x) = x/2$

🎯 PRACTICE QUESTIONS FOR CBSE BOARD EXAMS

Short Answer Questions (2-3 Marks)

Q1. Show that the relation R in the set R of real numbers, defined as R = {(a, b) : $a \leq b^2$} is neither reflexive nor symmetric nor transitive.

Q2. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1} is reflexive, symmetric or transitive.

Q3. Show that the function $f : R_* \rightarrow R_*$ defined by f(x) = 1/x is one-one and onto.

Long Answer Questions (5 Marks)

Q4. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by R = {(a, b) : |a - b| is even} is an equivalence relation. Find all equivalence classes.

Q5. Let A = R - {3} and B = R - {1}. Show that the function f : A → B defined by f(x) = (x - 2)/(x - 3) is one-one and onto, and find f^{-1}.

Q6. If f : R → R and g : R → R are defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, find fog and gof. Show that fog ≠ gof.

💡 TIPS FOR CBSE BOARD EXAMS 2025-26

✅ Do's for Board Exams:

- **For Relations:** Always check all three properties (reflexive, symmetric, transitive) separately when proving equivalence relations
- **Show Counter-examples:** Use specific counter-examples when disproving properties
- **For Functions:** Clearly show both one-one and onto proofs separately with proper steps
- **Inverse Functions:** Always verify $fof^{-1} = I$ and $f^{-1}of = I$ after finding inverse
- **Composition:** Remember to apply functions from right to left in gof: first f, then g
- **Write All Steps:** Show every algebraic step clearly for full marks
- **Use Proper Notation:** Use mathematical symbols correctly (∈, ⊆, ⇒, ∀, ∃)

⚠️ **IMPORTANT: Common Mistakes to Avoid:**

❌ Not checking all three properties for equivalence relations
❌ Confusing gof with fog - they are usually different!
❌ Not verifying inverse function properly
❌ Assuming a function is onto without proper proof
❌ Missing domain/co-domain restrictions
❌ Not writing "therefore" or "hence" for conclusions

📝 Marking Scheme Pattern (Expected):

- **2 Marks:** Check one property OR simple function verification
- **3 Marks:** Prove equivalence relation (all three properties)
- **4 Marks:** Equivalence relation + equivalence classes

- **5 Marks:** Bijective proof + find inverse function

📋 QUICK REVISION - KEY FORMULAS & RESULTS

RELATIONS

- Reflexive: (a, a) ∈ R ∀ a ∈ A
- Symmetric: (a, b) ∈ R ⇒ (b, a) ∈ R
- Transitive: (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R
- Equivalence Relation = Reflexive + Symmetric + Transitive

FUNCTIONS

- One-One: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- Onto: Range = Co-domain
- Bijective = One-One + Onto
- Invertible ⟺ Bijective
- (gof)(x) = g(f(x))
- gof ≠ fog (in general)
- $fof^{-1} = I$ and $f^{-1}of = I$

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"Pure mathematics is, in its way, the poetry of logical ideas."

— Albert Einstein

This study material is prepared strictly according to CBSE Board Exam Guidelines 2025-26

✅ NCERT Aligned | ✅ Board Exam Pattern | ✅ Comprehensive Coverage